

# Weak productions of new charmonium in semi-leptonic decays of $B_c$

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## Abstract

We study the weak productions of novel heavy mesons, such as  $\eta'_c$ ,  $h_c$ ,  $h'_c$ ,  $\chi'_{c0}$ ,  $X(3940)$ ,  $Y(3940)$ ,  $X(3872)$ , and  $Y(4260)$ , in the semi-leptonic  $B_c$  decays. Since there is still no definite answer for the components of  $X(3940)$ ,  $Y(3940)$ ,  $X(3872)$ ,  $Y(4260)$  so far, we will assign them as excited charmonium states with the possible quantum numbers constrained by the current experiments. As for the weak transition form factors, we calculate them in the framework of light-cone QCD sum rules approach, which is proved to be a powerful tool to deal with the non-perturbative hadronic matrix element. Our results indicate that different interpretations of  $X(3940)$  can result in remarkable discrepancy of the production rate in the  $B_c$  decays, which would help to clarify the inner structure of the  $X(3940)$  with the forthcoming LHC-b experiments. Besides, the predicted large weak production rates of  $X(3872)$  and  $Y(3940)$  in  $B_c$  decays and the small semi-leptonic decay rate for  $B_c \rightarrow Y(4260)$  all depend on their quantum number  $J^{PC}$  assignments. Moreover, the  $S - D$  mixing of various vector charmonium states in the weak decay of  $B_c$  is also discussed in this work. The future experimental measurements of these decays will test the inner structures of these particles, according to our predictions here.

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## I. INTRODUCTION

A number of new hidden charm states were observed recently by experiments, such as  $X(3872)$ ,  $X(3940)$ ,  $Y(3940)$ ,  $Z(3930)$  and  $Y(4260)$  [1, 2, 3, 4, 5]. Their quark structures are still not fully understood [6]. In particular, the  $X(3872)$ , which exhibits various impenetrable aspects, is labeled as the poster boy of the new heavy hadrons [7]. Although the quantum numbers  $J^{PC} = 1^{++}$  of  $X(3872)$  are strongly favored by the experiments, there is not a definite answer on its components yet due to the fact that none of the interpretations can fit all the available experiments satisfactorily. The assignment of  $X(3872)$  as a  $2^3P_1$  charmonium state, even without the mass gap problem<sup>1</sup> as claimed by calculations based on the Lattice QCD recently in [8], also bears other difficulties. The tiny decay width of  $X(3872)$ , whose upper bound is 2.3 MeV with 90% confidence level, is much less than the number predicted in theory [7]. Another puzzle is the G parity violation indicated by the measurement of the ratio of branching fractions  $\frac{BR(X \rightarrow J/\psi \pi^+ \pi^- \pi^0)}{BR(X \rightarrow J/\psi \pi^+ \pi^-)} = 1.0 \pm 0.4 \pm 0.3$  [9, 10, 11]. The difficulties of the charmonium interpretation invoke various models for the structure of  $X(3872)$ , such as multi-quark state [12, 13], hybrid meson [14], nuclear-like molecular state [15, 16, 17, 18, 19] and so on. In one word, the inner structure of  $X(3872)$  is still not settled down.

In addition to the intriguing particle  $X(3872)$ , other heavy hidden charm mesons  $X/Y(3940)$ ,  $Z(3930)$  and  $Y(4260)$  mentioned above also attract comprehensive attention recently [7], among which  $Z(3930)$  can be well established as the first radial excited states of tensor charmonium  $\chi_{c2}$  reasonably and will be left out in this paper. Even though the experimental results of  $h_c$  and  $\eta'_c$  are essentially consistent with theoretical expectations, there are still some particular aspects deserving further investigations [20, 22, 23, 24]. Besides, we also predict the production rate of  $h'_c$  state in the weak  $B_c$  decays, which has not discovered. In addition, the  $2S - 1D$  mixing of  $\psi(3686)$  and  $\psi(3770)$ , which is of great interest in quarkonium physics, is considered in the weak decays of  $B_c$ . More important, we also investigate the production of  $Y(4260)$  and  $\psi(4415)$  in the weak  $B_c$  decays as the mixing of  $4S$  and  $3D$  states. For the completeness, the  $3S - 2D$  mixing of  $\psi(4040)$  and  $\psi(4160)$  in the  $B_c$  decay is also included.

In this work, we do not attempt to discuss all the explanations of these states. Instead, we concentrate on the assignment of these heavy mesons as charmonium states with the possible quantum numbers constrained by the available experiments and then study their production properties in the  $B_c$  decays. To be more specific, we will assign the  $X(3872)$  as a  $2^3P_1$  charmonium, the  $Y(4260)$  as a  $4^3S_1$  charmonium, the  $X(3940)$  being either  $3^1S_0$  or  $2^3P_1$  charmonium, and  $2^3P_0(c\bar{c})$  state for the  $Y(3940)$ , the quantum

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<sup>1</sup> The mass of  $2^3P_1$  charmonium predicted by the quark model is about 100 MeV larger than the measured  $X(3872)$ .

numbers of which as the charmonium states are most favored by the current experiments [20, 21], although there is overpopulation of  $2^3P_1(c\bar{c})$  meson in the charmonium family. The non-leptonic weak decays  $B^+ \rightarrow X(3872)K^+$  and  $B_c \rightarrow X(3872)\pi(K)$  have been studied in [25, 26] in order to pry information of the inner structure of  $X(3872)$ . It is found there that different  $J^{PC}$  assignment of  $X(3872)$  will give quite different decay rate for future experiments to measure. Here we will focus on the semi-leptonic weak production of charmonium particles in  $B_c$  decays, where it is theoretically easier compared with that of non-leptonic decays. We will see that the different assignments of  $J^{PC}$  quantum number to  $X(3943)$  will give remarkable different branching ratios of semi-leptonic decays. Therefore our predictions can be used by future experiments to test the quark structures of these mesons. The main job of calculating the branching fractions of the semi-leptonic decays of  $B_c$  is to properly evaluate the hadronic matrix elements for  $B_c \rightarrow M_{c\bar{c}}$  ( $M$ = pseudo-scalar ( $P$ ), scalar ( $S$ ), vector ( $V$ ) or axial vector ( $A$ ) charmonium), namely the transition form factors.

The precise calculations of form factors are very complicated due to the non-perturbative QCD effects in the hadron as a bound state. Several methods have been developed to deal with this problem on the market so far, such as simple quark model [27], light-front approach [28, 29, 30], QCD sum rules (SVZ) [31, 32], light-cone QCD sum rules [33, 34, 35], perturbative QCD factorization approach [36, 37]. Although the QCD sum rules approach has made a big success, short distance expansion fails in non-perturbative condensate when applying the three-point sum rules to the computations of form factors in the large momentum transfer or large mass limit of heavy meson decays. The light-cone QCD sum rules, as a marriage of QCD sum rules techniques and the theory of hard exclusive processes, were developed in an attempt to overcome the difficulties [38] involved in the SVZ sum rules. The basic idea of light-cone QCD sum rules [28, 29, 30, 38, 39] is to adopt the twist expansion of correlation functions near the light-cone instead of the dimension expansion of operators at short distance. Therefore, the essential inputs in the light-cone QCD sum rules is the hadronic distribution amplitudes other than vacuum condensates in the QCD sum rules. One important advantage of light-cone QCD sum rules is that it allows a systematic inclusion of both hard scattering effects and soft contributions [39]. In view of the above arguments, we will estimate the form factors for  $B_c$  to charmonium states based on the light-cone QCD sum rules approach in this work.

The structure of this paper is organized as follows: we first display the light-cone distribution amplitudes of various charmonium states in section II. The light-cone QCD sum rules for the form factors responsible for the decay modes  $B_c \rightarrow M_{c\bar{c}}$  are derived in section III. The numerical computations of form factors in light-cone QCD sum rules are performed in section IV. The decay rates for semileptonic decays of  $B_c$  to various charmonium states, a brief analysis on comparisons with the results that obtained

with the help of other approaches in the literature and discussions on the S-D mixing of  $\psi(3686)$  and  $\psi(3770)$  in the weak decay of  $B_c$  are also included in this section. The last section is devoted to our conclusion.

## II. THE LIGHT-CONE DISTRIBUTION AMPLITUDES OF CHARMONIUM STATES

The light-cone distribution amplitudes (LCDAs) of pseudoscalar charmonium can be defined by the following non-local matrix element [40]

$$\langle P(p) | \bar{c}(x)_\alpha c(0)_\beta | 0 \rangle = -\frac{i}{4} f_P \int_0^1 du e^{iup \cdot x} [(\gamma_5 \not{p})_{\beta\alpha} \phi^v(u) + m_P (\gamma_5)_{\beta\alpha} \phi^s(u)], \quad (1)$$

where  $\phi^v(u)$  and  $\phi^s(u)$  are twist-2 and twist-3 LCDAs of the pseudoscalar charmonium respectively. The decay constant  $f_P$  can be determined generally by decay width of the double photons decay of the pseudoscalar meson as [41]

$$\Gamma(P \rightarrow \gamma\gamma) = \frac{4(4\pi\alpha)^2 f_P^2}{81\pi m_P}. \quad (2)$$

Making use of the branching fractions of  $\eta_c \rightarrow \gamma\gamma$  and the full width of  $\eta_c$  [42]

$$\text{BR}(\eta_c \rightarrow \gamma\gamma) = (2.8 \pm 0.9) \times 10^{-4}, \quad \Gamma_{\eta_c} = (25.5 \pm 3.4) \text{MeV}, \quad (3)$$

we can achieve the decay constant  $f_{\eta_c}$  as  $401_{-76}^{+65} \text{MeV}$ . However, there is no data on  $\eta_c'' \rightarrow \gamma\gamma$  till now, hence it is impossible to extract the decay constant of  $\eta_c''$  directly from the experiments. In view of this point, we fix the decay constant  $f_{\eta_c''}$  through the assumption,  $\frac{f_{\eta_c''}}{f_{\eta_c}} = \frac{f_{\psi''}}{f_{J/\psi}}$ , which has been used in [43] before. The decay constant of vector charmonium can be derived through leptonic decay  $V \rightarrow e^+e^-$  as

$$f_V = \sqrt{\frac{3m_V \Gamma_{V \rightarrow ee}}{4\pi\alpha^2 Q_c^2}}. \quad (4)$$

Combining the above relation and the data given in [42]

$$\Gamma_{J/\psi \rightarrow ee} = (5.55 \pm 0.14 \pm 0.02) \text{keV}, \quad \Gamma_{\psi' \rightarrow ee} = (2.48 \pm 0.06) \text{keV}, \quad \Gamma_{\psi'' \rightarrow ee} = (0.86 \pm 0.07) \text{keV}, \quad (5)$$

we can obtain the decay constants of  $\psi(nS)$  ( $n = 1, 2, 3$ ) as

$$f_{J/\psi} = 416_{-6}^{+5} \text{MeV}, \quad f_{\psi'} = 304_{-4}^{+3} \text{MeV}, \quad f_{\psi''} = 187 \pm 8 \text{MeV}. \quad (6)$$

In light of the assumption mentioned above, we arrive at the decay constant of  $\eta_c''$  as  $180_{-32}^{+27} \text{MeV}$ . Moreover, the decay constant of  $\eta_c'$  can be determined as  $293_{-56}^{+48} \text{MeV}$ .

It needs to be pointed out that the tensor structure, which is suppressed in the heavy quark limit, has been neglected in the right hand side of the Eq. (1). When it comes to the explicit forms of  $\phi^v(u)$  and

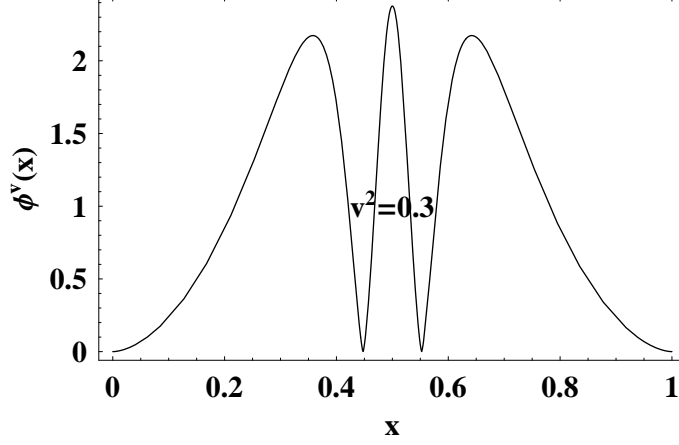


FIG. 1: The shape of the distribution amplitude  $\phi^v(x)$  for  $\eta_c''$  with  $v^2 = 0.3$ .

$\phi^s(u)$ , we will adopt a simple model advocated in [45]. Firstly, one should write down the Schrödinger equal-time wave function  $\Psi_{Sch}(r)$  for the Coulomb potential, and then perform the Fourier transformation of it to the momentum space as  $\Psi_{Sch}(k)$ . Next, in terms of the substitution assumption proposed in [46] (see also Eq. (A3)), we can derive the expression of wave function  $\Psi_{Sch}(x_i, \mathbf{k}_\perp)$  from  $\Psi_{Sch}(k)$ , where the momentum fractions  $x_1, x_2$  of  $c$  and  $\bar{c}$  quarks in the charmonium satisfy the relation  $x_1 + x_2 = 1$ . Finally, one can achieve at the LCDAs of charmonium  $\Psi_{Sch}(x_i)$  by integrating over the transverse momentum  $\mathbf{k}_\perp$ . Based on this prescription, we can obtain the LCDAs for  $\eta_c''$  as

$$\begin{aligned}\phi^v(x) &= 10.8x(1-x) \left\{ \frac{x(1-x)[1-4x(1-x)(1+\frac{v^2}{27})]^2}{[1-4x(1-x)(1-\frac{v^2}{9})]^3} \right\}^{1-v^2}, \\ \phi^s(x) &= 2.1 \left\{ \frac{x(1-x)[1-4x(1-x)(1+\frac{v^2}{27})]^2}{[1-4x(1-x)(1-\frac{v^2}{9})]^3} \right\}^{1-v^2},\end{aligned}\quad (7)$$

where the variable  $v$  reflects the mean charm quark velocity and is taken as  $v^2 = 0.30 \pm 0.05$  [45] in the numerical analysis. To be more clear, the shape of the distribution amplitude  $\phi^v(x)$  is shown in Fig. 1 with  $v^2 = 0.3$ .

Similarly, we can also derive the LCDAs for  $\eta_c'$

$$\begin{aligned}\phi^v(x) &= 10.6x(1-x) \left\{ \frac{x(1-x)(1-2x)^2}{[1-4x(1-x)(1-\frac{v^2}{4})]^2} \right\}^{1-v^2}, \\ \phi^s(x) &= 2.1 \left\{ \frac{x(1-x)(1-2x)^2}{[1-4x(1-x)(1-\frac{v^2}{4})]^2} \right\}^{1-v^2}.\end{aligned}\quad (8)$$

Similarly, the LCDAs of scalar charmonium state can be defined by

$$\langle S(p) | \bar{c}(x)_\alpha c(0)_\beta | 0 \rangle = \frac{1}{4} f_S \int_0^1 du e^{iup \cdot x} [(\not{p})_{\beta\alpha} \chi^v(u) + m_S(I)_{\beta\alpha} \chi^s(u)], \quad (9)$$

with  $\chi^v(u)$  and  $\chi^s(u)$  being the twist-2 and twist-3 DAs for the scalar meson respectively. Based on the method of building the model for heavy quarkonium's distribution amplitudes given above, we can obtain the explicit forms of distribution amplitudes as

$$\begin{aligned}\chi^v(x) &= 90.2x(1-x)(1-2x)\left\{\frac{x(1-x)(1-2x)^4}{[1-4x(1-x)(1-\frac{v^2}{9})]^3}\right\}^{1-v^2}, \\ \chi^s(x) &= 1.9\left\{\frac{x(1-x)(1-2x)^4}{[1-4x(1-x)(1-\frac{v^2}{9})]^3}\right\}^{1-v^2},\end{aligned}\quad (10)$$

for the  $2^3P_0$  charmonium  $\chi'_{c0}$ . In addition, the decay constant of  $\chi'_{c0}$  can be calculated as  $263^{+6}_{-7}\text{MeV}$  making use of the assumption  $\frac{f_{\chi'_{c0}}}{f_{\chi_{c0}}} = \frac{f_{\psi'}}{f_{J/\psi}}$  and the value of  $f_{\chi_{c0}}$ , which was estimated to be  $360\text{MeV}$  in Ref. [40, 44].

The non-local matrix element associating with the vector charmonium can be decomposed as [45]

$$\begin{aligned}\langle V(p, \epsilon) | \bar{c}(x)_\alpha c(0)_\beta | 0 \rangle &= \frac{1}{4} \int_0^1 du e^{iup \cdot x} \left\{ f_V m_V (\not{\epsilon}^* - \frac{\epsilon^* \cdot x}{p \cdot x} \not{p}) V_\perp(u) + f_V m_V \frac{\epsilon^* \cdot x}{p \cdot x} \not{p} V_L(u) \right. \\ &\quad \left. + f_V^T \not{\epsilon} \not{p} V_T(u) + \frac{1}{4} (f_V - \frac{2m_c}{m_V} f_V^T) \epsilon_{\mu\nu\alpha\beta} \gamma^\mu \gamma^5 \epsilon^{*\nu} p^\alpha x^\beta V_A(u) \right\}_{\beta\alpha},\end{aligned}\quad (11)$$

where  $V_L(u)$ ,  $V_T(u)$  are the leading twist longitudinal and transverse LCDAs of vector charmonium, and  $V_\perp(u)$ ,  $V_A(u)$  are the twist-3 ones. Following the methods described above, we can deduce the manifest expressions of these distribution amplitudes as

$$\begin{aligned}V_L(x) &= 10.8x(1-x)\left\{\frac{x(1-x)((1-2x)^2[1-4x(1-x)(1+\frac{v^2}{16})]^2)}{[1-4x(1-x)(1-\frac{v^2}{16})]^4}\right\}^{1-v^2}, \\ V_\perp(x) &= 1.74[1+(1-2x)^2]\left\{\frac{x(1-x)((1-2x)^2[1-4x(1-x)(1+\frac{v^2}{16})]^2)}{[1-4x(1-x)(1-\frac{v^2}{16})]^4}\right\}^{1-v^2} \\ V_T(x) &= V_A(x) = V_L(x),\end{aligned}\quad (12)$$

for the  $4^3S_1$  charmonium. In the numerical calculations, the decay constants  $f_V$  and  $f_V^T$  are assumed to be equal [40]. The purely leptonic decay of  $Y(4260) \rightarrow e^+e^-$  is estimated to be  $0.72\text{keV}$  [21], from which we can obtain the decay constant  $f_{Y(4260)}$  as  $176\text{MeV}$ .

Similarly, the light-cone distribution amplitudes of  $3^3D_1$  charmonium can be derived as

$$\begin{aligned}V_L(x) &= 2.8x(1-x)\left\{\frac{x^2(1-x)^2(1-2x)^6}{[1-4x(1-x)(1-\frac{v^2}{25})]^5}\right\}^{1-v^2}, \\ V_\perp(x) &= 0.62[1+(1-2x)^2]\left\{\frac{x^2(1-x)^2(1-2x)^6}{[1-4x(1-x)(1-\frac{v^2}{25})]^5}\right\}^{1-v^2}, \\ V_T(x) &= V_A(x) = V_L(x).\end{aligned}\quad (13)$$

With the hypothesis  $\frac{f_{\psi(3^3D_1)}}{f_{\psi(1^3D_1)}} = \frac{f_{\psi''}}{f_{J/\psi}}$  and  $f_{\psi(1^3D_1)} = 47.8\text{ MeV}$  [47], we can achieve the value of  $f_{\psi(3^3D_1)}$  as  $21.5^{+0.6}_{-0.5}\text{MeV}$  under the above assumption. This is a quite small decay constant, comparing with that

of the corresponding S-wave charmonium states. This will surely lead to the quite small form factors, since the transition form factors are proportion to the decay constant of the final state meson as can be observed from the light-cone sum rules in the next section. In the same way, we can arrive at the decay constant of  $\psi(2^3D_1)$  as  $f_{\psi(2^3D_1)} = 34.9^{+0.8}_{-0.9}\text{MeV}$ .

For the sake of investigating the  $2S - 1D$  mixing of  $\psi(3686)$  and  $\psi(3770)$ , it is necessary to derive the light-cone distribute amplitudes for the  $\psi(2^3S_1)$  and  $\psi(1^3D_1)$  based on the model discussed above. To be more specific, the LCDAs for  $\psi(2^3S_1)$  can be given by

$$\begin{aligned} V_L(x) &= V_T(x) = V_A(x) = 10.6x(1-x) \left\{ \frac{x(1-x)(1-2x)^2}{[1-4x(1-x)(1-\frac{v^2}{4})]^2} \right\}^{1-v^2}, \\ V_\perp(x) &= 1.7[1+(1-2x)^2] \left\{ \frac{x(1-x)(1-2x)^2}{[1-4x(1-x)(1-\frac{v^2}{4})]^2} \right\}^{1-v^2}; \end{aligned} \quad (14)$$

while the LCDAs for  $\psi(1^3D_1)$  can read as

$$\begin{aligned} V_L(x) &= V_T(x) = V_A(x) = 3.6x(1-x) \left\{ \frac{x^2(1-x)^2(1-2x)^2}{[1-4x(1-x)(1-\frac{v^2}{9})]^3} \right\}^{1-v^2}, \\ V_\perp(x) &= 0.77[1+(1-2x)^2] \left\{ \frac{x^2(1-x)^2(1-2x)^2}{[1-4x(1-x)(1-\frac{v^2}{9})]^3} \right\}^{1-v^2}. \end{aligned} \quad (15)$$

Moreover, we also would like to present the explicit forms of LCDAs for  $\psi(3^3S_1)$  and  $\psi(2^3D_1)$ , which are essential to study the  $3S - 2D$  mixing of  $\psi(4040)$  and  $\psi(4160)$ . The LCDAs for  $\psi(3^3S_1)$  can be calculated as

$$\begin{aligned} V_L(x) &= V_T(x) = V_A(x) = 10.8x(1-x) \left\{ \frac{x(1-x)[1-4x(1-x)(1+\frac{v^2}{27})]^2}{[1-4x(1-x)(1-\frac{v^2}{9})]^3} \right\}^{1-v^2}, \\ V_\perp(x) &= 1.7[1+(1-2x)^2] \left\{ \frac{x(1-x)[1-4x(1-x)(1+\frac{v^2}{27})]^2}{[1-4x(1-x)(1-\frac{v^2}{9})]^3} \right\}^{1-v^2}; \end{aligned} \quad (16)$$

while the LCDAs for  $\psi(2^3D_1)$  are given by

$$\begin{aligned} V_L(x) &= V_T(x) = V_A(x) = 3.2x(1-x) \left\{ \frac{x^2(1-x)^2(1-2x)^4}{[1-4x(1-x)(1-\frac{v^2}{16})]^4} \right\}^{1-v^2}, \\ V_\perp(x) &= 0.70[1+(1-2x)^2] \left\{ \frac{x^2(1-x)^2(1-2x)^4}{[1-4x(1-x)(1-\frac{v^2}{16})]^4} \right\}^{1-v^2}. \end{aligned} \quad (17)$$

As far as the axial-vector charmonium is concerned, the corresponding non-local matrix element can be analyzed as [48]

$$\begin{aligned} \langle A(p, \epsilon) | \bar{c}(x)_\alpha c(0)_\beta | 0 \rangle &= -\frac{i}{4} \int_0^1 du e^{iup \cdot x} \left\{ f_A m_A (\not{\epsilon}^* - \frac{\epsilon^* \cdot x}{p \cdot x} \not{p}) \gamma_5 g_\perp^{(a)}(u) + f_A m_A \frac{\epsilon^* \cdot x}{p \cdot x} \not{p} \gamma_5 \phi_\parallel(u) \right. \\ &\quad \left. + f_A^T \not{\epsilon} \not{p} \gamma_5 \phi_\perp(u) + \frac{1}{4} (f_A - \frac{2m_c}{m_A} f_A^T) \epsilon_{\mu\nu\alpha\beta} \gamma^\mu \epsilon^{*\nu} p^\alpha x^\beta g_\perp^v(u) \right\}_{\beta\alpha}, \end{aligned} \quad (18)$$

where  $\phi_{\parallel}(u)$ ,  $\phi_{\perp}(u)$  are of twist-2, and  $g_{\perp}^v(u)$  and  $g_{\perp}^a(u)$  are the twist-3 LCDAs of axial-vector charmonium. As for the  $n^3P_1$  states,  $\phi_{\parallel}(u)$ ,  $g_{\perp}^v(u)$  and  $g_{\perp}^a(u)$  are symmetric under the exchange of momentum fractions  $u$  and  $1-u$ , but  $\phi_{\perp}(u)$  is anti-symmetric under this exchange. On the contrary,  $\phi_{\perp}(u)$  is symmetric for  $n^1P_1$  states, while  $\phi_{\parallel}(u)$ ,  $g_{\perp}^v(u)$  and  $g_{\perp}^a(u)$  are anti-symmetric in this case. Following the procedure of constructing the wave functions for heavy quarkonium shown above, we can arrive at

$$\phi_{\perp}(x) = 90.2x(1-x)(1-2x)\left\{\frac{x(1-x)(1-2x)^4}{[1-4x(1-x)(1-\frac{v^2}{9})]^3}\right\}^{1-v^2}, \quad (19)$$

for the  $2^3P_1$  charmonium,

$$\phi_{\perp}(x) = 10.6x(1-x)\left\{\frac{x(1-x)(1-2x)^2}{[1-4x(1-x)(1-\frac{v^2}{4})]^2}\right\}^{1-v^2}, \quad (20)$$

for the  $1^1P_1$  charmonium  $h_c$ , and

$$\phi_{\perp}(x) = 9.5x(1-x)\left\{\frac{x(1-x)(1-2x)^4}{[1-4x(1-x)(1-\frac{v^2}{9})]^3}\right\}^{1-v^2}, \quad (21)$$

for the  $2^1P_1$  charmonium  $h'_c$ .

As can be seen below, only the leading twist LCDA of axial-vector charmonium  $\phi_{\perp}(u)$  is involved in the light-cone QCD sum rules of form factors, hence, the expressions of other three distribution amplitudes will not be shown here. Notice that we assume the decay constants  $f_A = f_A^T$  in the practical numerical analysis, the same as that for the vector charmonium. Meanwhile, the decay constant of  $P$  wave charmonium  $2^3P_1(c\bar{c})$ , was estimated to be 207MeV [49] very recently. Moreover, the decay constant of  $h_c$  and  $h'_c$  are taken the same as that for the  $\chi_{c1}$  in [50] and  $\chi'_{c1}$  respectively, namely  $f_{h_c} = f_{\chi_{c1}} = 335\text{MeV}$  [40],  $f_{h'_c} = f_{\chi'_{c1}} = 207\text{MeV}$ .

### III. LIGHT-CONE QCD SUM RULES FOR THE WEAK TRANSITION FORM FACTORS

For the semi-leptonic decays of  $B_c \rightarrow M_{c\bar{c}}l\bar{\nu}_l$ , the effective weak Hamiltonian is given by

$$\mathcal{H}_{eff}(b \rightarrow cl\bar{\nu}_l) = \frac{G_F}{\sqrt{2}}V_{cb}\bar{c}\gamma_{\mu}(1-\gamma_5)b\bar{l}\gamma^{\mu}(1-\gamma_5)\nu_l + h.c., \quad (22)$$

where  $V_{cb}$  is the corresponding Cabbibo-Kobayashi-Maskawa (CKM) matrix element. In order to estimate the decay rates of  $B_c \rightarrow M_{c\bar{c}}l\bar{\nu}_l$ , we need to calculate the hadronic matrix element  $\langle M_{c\bar{c}}|\bar{c}\gamma_{\mu}(1-\gamma_5)b|B_c\rangle$  at first, which can be conventionally parameterized in the following forms:

$$\langle P_{c\bar{c}}(p)|\bar{c}\gamma_{\mu}b|B_c(p+q)\rangle = f_+(q^2)p_{\mu} + f_-(q^2)q_{\mu}, \quad (23)$$

$$\langle S_{c\bar{c}}(p)|\bar{c}\gamma_{\mu}\gamma_5b|B_c(p+q)\rangle = -i[f_+(q^2)p_{\mu} + f_-(q^2)q_{\mu}], \quad (24)$$



$$\begin{aligned}\langle V_{c\bar{c}}(p)|\bar{c}\gamma_\mu(1-\gamma_5)b|B_c(p+q)\rangle &= \frac{2V(q^2)}{m_{B_c}+m_V}\epsilon_{\mu\alpha\beta\gamma}\epsilon^{*\alpha}q^\beta p^\gamma - i(m_{B_c}+m_V)A_1(q^2)\epsilon_\mu^* \\ &+ \frac{iA_2(q^2)}{m_{B_c}+m_V}(\epsilon^*\cdot q)(2p+q)_\mu + \frac{iA_3(q^2)}{m_{B_c}+m_V}(\epsilon^*\cdot q)q_\mu,\end{aligned}\quad (25)$$

$$\begin{aligned}\langle A_{c\bar{c}}(p)|\bar{c}\gamma_\mu(1-\gamma_5)b|B_c(p+q)\rangle &= -\frac{2iA(q^2)}{m_{B_c}-m_A}\epsilon_{\mu\alpha\beta\gamma}\epsilon^{*\alpha}q^\beta p^\gamma - (m_{B_c}-m_A)V_1(q^2)\epsilon_\mu^* \\ &+ \frac{V_2(q^2)}{m_{B_c}-m_A}(\epsilon^*\cdot q)(2p+q)_\mu + \frac{V_3(q^2)}{m_{B_c}-m_A}(\epsilon^*\cdot q)q_\mu,\end{aligned}\quad (26)$$

where the anti-symmetric fourth rank tensor is defined as  $\text{Tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5] = 4i\epsilon_{\mu\nu\rho\sigma}$ .

Below, we will derive the general formulae for the form factors of  $B_c \rightarrow M_{c\bar{c}} (M = P, S, V, A)$  in the light-cone QCD sum rules approach. Following Ref. [51, 52], the correlation function is selected with the insertion of chiral current, to which the twist-3 distribution amplitudes of final states do not contribute at all for semi-leptonic  $B_c \rightarrow P(S)$  decays. As for the  $B_c \rightarrow V(A)$  decays, the two-particle distribution amplitudes of twist-3 also have no effect on the correlation function with the insertion of chiral current in the heavy charm quark mass limit. Besides, the twist-3 distribution amplitudes relating to the three-particle  $\bar{c}cg$  Fock state, which are suppressed by a factor  $(\Lambda_{QCD}/m_{\bar{c}c})^2$  [45] with  $m_{\bar{c}c}$  being the mass of the charmonium, are also omitted in this work. The estimation of correlation functions in the QCD representation can be carried out following the standard prescription given in [53, 54].

#### A. Light-cone QCD sum rules for the weak transition form factors of $B_c \rightarrow P_{c\bar{c}}$

Based on the above analysis, we firstly construct the following correlator  $\Pi_\mu(p, q)$  with the insertion of the chiral current:

$$\Pi_\mu(p, q) = i \int d^4x e^{iq\cdot x} \langle P_{c\bar{c}}(p) | T\{\bar{c}(x)\gamma_\mu(1+\gamma_5)b(x), \bar{b}(0)i(1+\gamma_5)c(0)\} | 0 \rangle. \quad (27)$$

One character of this correlation function is that twist-3 distribution amplitude of pseudoscalar charmonium has no influence on it and therefore the theoretical uncertainties can be reduced considerably in this way. Inserting the complete sets of hadronic states with the quantum numbers the same as  $B_c$  and making use of the following definition

$$\langle B_c | \bar{b}i(1+\gamma_5)c | 0 \rangle = \frac{m_{B_c}^2 f_{B_c}}{m_b + m_c}, \quad (28)$$

we can arrive at the hadronic representation of correlation function (27) as below:

$$\begin{aligned}\Pi_\mu(p, q) &= \frac{\langle P_{c\bar{c}}(p) | \bar{c}\gamma_\mu(1+\gamma_5)b | B_c(p+q) \rangle \langle B_c(p+q) | \bar{b}i(1+\gamma_5)c | 0 \rangle}{m_{B_c}^2 - (p+q)^2} \\ &+ \sum_h \frac{\langle P_{c\bar{c}}(p) | \bar{c}\gamma_\mu(1+\gamma_5)b | h(p+q) \rangle \langle h(p+q) | \bar{b}i(1+\gamma_5)c | 0 \rangle}{m_h^2 - (p+q)^2}\end{aligned}$$

$$= \frac{m_{B_c}^2 f_{B_c} (f_+(q^2) p_\mu + f_-(q^2) q_\mu)}{(m_b + m_c)(m_{B_c}^2 - (p + q)^2)} + \int_{s_0^{B_c}}^\infty ds \frac{\rho_+^h(s, q^2) p_\mu + \rho_-^h(s, q^2) q_\mu}{s - (p + q)^2}, \quad (29)$$

where we have expressed the contributions from higher states of the  $B_c$  channel in the form of dispersion integral with  $s_0^{B_c}$  being the threshold parameter corresponding to the  $B_c$  channel. On the other hand, we can also calculate the correlation function at the quark level:

$$\begin{aligned} \Pi_\mu(p, q) &= \Pi_+^{QCD}(q^2, (p + q)^2) p_\mu + \Pi_-^{QCD}(q^2, (p + q)^2) q_\mu \\ &= \int_{(m_b + m_c)^2}^\infty ds \frac{1}{\pi} \frac{\text{Im} \Pi_+^{QCD}(s, q^2)}{s - (p + q)^2} p_\mu + \int_{(m_b + m_c)^2}^\infty ds \frac{1}{\pi} \frac{\text{Im} \Pi_-^{QCD}(s, q^2)}{s - (p + q)^2} q_\mu. \end{aligned} \quad (30)$$

Utilizing the quark-hadron duality assumption

$$\rho_i^h(s, q^2) = \frac{1}{\pi} \text{Im} \Pi_i^{QCD}(s, q^2) \Theta(s - s_0^h), \quad (31)$$

with  $i = “+,” -”$  and performing the Borel transformation

$$\hat{\mathcal{B}}_{M^2} = \lim_{\substack{-(p+q)^2, n \rightarrow \infty \\ -(p+q)^2/n = M^2}} \frac{(-(p+q)^2)^{(n+1)}}{n!} \left( \frac{d}{d(p+q)^2} \right)^n, \quad (32)$$

with variable  $(p + q)^2$  to both two representations of the correlation function, we can finally derive the sum rules for the form factors

$$f_i(q^2) = \frac{m_b + m_c}{\pi f_{B_c} m_{B_c}^2} \int_{(m_b + m_c)^2}^{s_0^{B_c}} \text{Im} \Pi_i^{QCD}(s, q^2) \exp\left(\frac{m_{B_c}^2 - s}{M^2}\right) ds. \quad (33)$$

The QCD representation of correlation function (27) can be calculated in terms of operator product expansion (OPE) in both of the large space-like region  $(p + q)^2 \ll -(m_b + m_c)^2$  and the low momentum transfer region [54, 55]  $q^2 \leq (m_b - m_c)^2 - 2\Lambda_{QCD}(m_b - m_c) \simeq 8.2 \text{ GeV}^2$ , where the value of  $\Lambda_{QCD}$  is usually taken as 0.5 GeV. It is expected the light-cone QCD sum rules approach for the transition form factors will break down at large momentum transfer [54], since the light-cone expansion for the description of final state meson is not well-pleasing in this case and the contributions from the higher twists would be important. The leading order contribution in the OPE can be gained simply by contracting the b-quark operators in the correlator (27) to a free b-quark propagator

$$\langle 0 | b(x) \bar{b}(0) | 0 \rangle = \int \frac{d^4 k}{i(2\pi)^4} e^{-ik \cdot x} \frac{\not{k} + m_b}{m_b^2 - k^2}, \quad (34)$$

which can be represented by Fig. 2 intuitively.

Then we arrive at the correlation function at the quark level as

$$\begin{aligned} \Pi_+(q^2, (p + q)^2) &= -2m_b f_{P_{c\bar{c}}} \int_0^1 du \frac{\phi^v(u)}{(q + up)^2 - m_b^2 + i\epsilon} + \text{contributions from higher twists}, \\ \Pi_-(q^2, (p + q)^2) &= 0 + \text{contributions from higher twists}, \end{aligned} \quad (35)$$

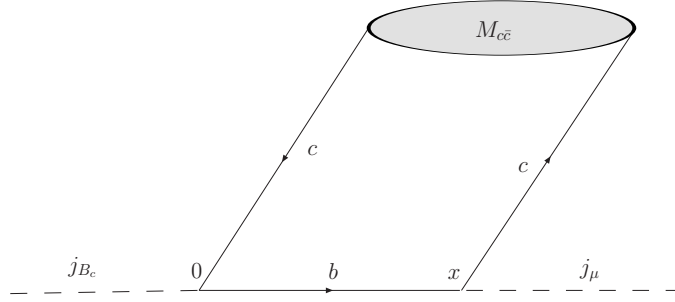


FIG. 2: The tree level contribution to the correlation function Eq. (27), where the current  $j_{B_c}(0)$  describe the  $B_c$  channel and the current  $j_{\mu}(x)$  is associate with the  $b \rightarrow c$  transition.

where the higher twists contributions are at least from twist-4 distribution amplitudes of the pseudoscalar charmonium [52, 55, 56]. Substituting the Eq.(35) to Eq.(33), we can finally derive the light-cone QCD sum rules for the form factors  $f_i(q^2)$  as below

$$\begin{aligned} f_+(q^2) &= \frac{2m_b(m_b + m_c)f_{P_{c\bar{c}}}\exp\left(\frac{m_{B_c}^2}{M^2}\right)}{f_{B_c}m_{B_c}^2} \int_{\Delta}^1 \frac{du}{u} \phi^v(u) \exp\left[-\frac{m_b^2 - \bar{u}(q^2 - up^2)}{uM^2}\right], \\ f_-(q^2) &= 0, \end{aligned} \quad (36)$$

up to the accuracy of twist-3 LCDAs, with

$$\Delta = \frac{-(s - q^2 - p^2) + \sqrt{(s - q^2 - p^2)^2 + 4p^2(m_b^2 - q^2)}}{2p^2}, \quad (37)$$

$p^2$  being the mass square for the corresponding charmonium state ( $m_P^2$  in this subsection) and  $s$  being the threshold value of  $B_c$  channel. It needs to be emphasized that the vanishing of  $f_-(q^2)$  up to the twist-3 LCDAs of pseudoscalar charmonium and leading order of the strong coupling constant  $\alpha_s$  is the consequence of the large-recoil symmetry [57], which emerges in the case of large recoil momentum for the final state meson and can be broken by the hard gluon corrections [58].

### B. Light-cone QCD sum rules for the weak transition form factors of $B_c \rightarrow S_{c\bar{c}}$

Following the derivation of the light-cone sum rules for  $B_c \rightarrow P_{c\bar{c}}$ , the correlation function of  $B_c \rightarrow S_{c\bar{c}}$  can be written as

$$\Pi_{\mu}(p, q) = i \int d^4x e^{iq \cdot x} \langle S_{c\bar{c}}(p) | T \{ \bar{c}(x) \gamma_{\mu} (1 + \gamma_5) b(x), \bar{b}(0) i (1 + \gamma_5) c(0) \} | 0 \rangle. \quad (38)$$

Matching the results of the above correlator calculated in the quark level and hadron representation respectively and performing Borel transformation with the variable  $(p + q)^2$ , we can achieve the light-

cone sum rules for the transition form factors as below

$$\begin{aligned} f_+(q^2) &= -\frac{2m_b(m_b + m_c)f_{S_{c\bar{c}}}}{f_{B_c}m_{B_c}^2}\exp\left(\frac{m_{B_c}^2}{M^2}\right)\int_{\Delta}^1 \frac{du}{u}\chi^v(u)\exp\left[-\frac{m_b^2 - \bar{u}(q^2 - up^2)}{uM^2}\right], \\ f_-(q^2) &= 0, \end{aligned} \quad (39)$$

where the lower limit of the integral  $\Delta$  has been given in Eq.(37).

### C. Light-cone QCD sum rules for the weak transition form factors of $B_c \rightarrow V_{c\bar{c}}$

In the same way, the correlation function with the insertion of chiral current for  $B_c \rightarrow V_{c\bar{c}}$  can be chosen as

$$\Pi_{\mu}(p, q) = i \int d^4x e^{iq \cdot x} \langle V_{c\bar{c}}(p) | T \{ \bar{c}(x) \gamma_{\mu} (1 - \gamma_5) b(x), \bar{b}(0) i(1 + \gamma_5) c(0) \} | 0 \rangle. \quad (40)$$

The hadronic representation of this correlator can be derived as

$$\begin{aligned} \Pi_{\mu}(p, q) &= \frac{\langle V_{c\bar{c}}(p) | \bar{c} \gamma_{\mu} (1 - \gamma_5) b | B_c(p+q) \rangle \langle B_c(p+q) | \bar{b} i(1 + \gamma_5) c | 0 \rangle}{m_{B_c}^2 - (p+q)^2} \\ &+ \sum_h \frac{\langle V_{c\bar{c}}(p) | \bar{c} \gamma_{\mu} (1 - \gamma_5) b | h(p+q) \rangle \langle h(p+q) | \bar{b} i(1 + \gamma_5) c | 0 \rangle}{m_h^2 - (p+q)^2} \\ &= \frac{2m_{B_c}^2 f_{B_c} V(q^2)}{(m_b + m_c)(m_{B_c} + m_{V_{c\bar{c}}})(m_{B_c}^2 - (p+q)^2)} \epsilon_{\mu\alpha\beta\gamma} \epsilon^{*\alpha} q^{\beta} p^{\gamma} - i \frac{m_{B_c}^2 f_{B_c} (m_{B_c} + m_{V_{c\bar{c}}}) A_1(q^2)}{(m_b + m_c)(m_{B_c}^2 - (p+q)^2)} \epsilon_{\mu}^* \\ &+ i \frac{m_{B_c}^2 f_{B_c} (\epsilon^* \cdot q) [A_2(q^2)(2p+q)_{\mu} + A_3(q^2)q_{\mu}]}{(m_b + m_c)(m_{B_c} + m_{V_{c\bar{c}}})(m_{B_c}^2 - (p+q)^2)} + \int_{s_0^{B_c}}^{\infty} ds \frac{\rho_V^h(s, q^2)}{s - (p+q)^2} \epsilon_{\mu\alpha\beta\gamma} \epsilon^{*\alpha} q^{\beta} p^{\gamma} \\ &+ \int_{s_0^{B_c}}^{\infty} ds \frac{\rho_{A_1}^h(s, q^2)}{s - (p+q)^2} \epsilon_{\mu}^* + \int_{s_0^{B_c}}^{\infty} ds \frac{\rho_{A_2}^h(s, q^2)(2p+q)_{\mu} + \rho_{A_3}^h(s, q^2)q_{\mu}}{s - (p+q)^2} (\epsilon^* \cdot q). \end{aligned} \quad (41)$$

Besides, the correlation function in Eq.(40) can also be formulated as

$$\begin{aligned} \Pi_{\mu}(p, q) &= \Pi_V^{QCD}(q^2, (p+q)^2) \epsilon_{\mu\alpha\beta\gamma} \epsilon^{*\alpha} q^{\beta} p^{\gamma} - i \Pi_{A_1}^{QCD}(q^2, (p+q)^2) \epsilon_{\mu}^* \\ &+ i \Pi_{A_2}^{QCD}(q^2, (p+q)^2) (\epsilon^* \cdot q) (2p+q)_{\mu} + i \Pi_{A_3}^{QCD}(q^2, (p+q)^2) (\epsilon^* \cdot q) q_{\mu} \\ &= \int_{(m_b+m_c)^2}^{\infty} ds \frac{1}{\pi} \frac{\text{Im} \Pi_V^{QCD}(s, q^2)}{s - (p+q)^2} \epsilon_{\mu\alpha\beta\gamma} \epsilon^{*\alpha} q^{\beta} p^{\gamma} - i \int_{(m_b+m_c)^2}^{\infty} ds \frac{1}{\pi} \frac{\text{Im} \Pi_{A_1}^{QCD}(s, q^2)}{s - (p+q)^2} \epsilon_{\mu}^* \\ &+ i \int_{(m_b+m_c)^2}^{\infty} ds \frac{1}{\pi} \frac{\text{Im} \Pi_{A_2}^{QCD}(s, q^2)(2p+q)_{\mu} + \text{Im} \Pi_{A_3}^{QCD}(s, q^2)q_{\mu}}{s - (p+q)^2} (\epsilon^* \cdot q). \end{aligned} \quad (42)$$

Matching these two representations of the correlator and performing the Borel transforming with the variable  $(p+q)^2$  on them, we can obtain the light-cone QCD sum rules for the form factors of  $B_c \rightarrow V_{c\bar{c}}$  as

$$V(q^2) = \frac{(m_b + m_c)(m_{B_c} + m_{V_{c\bar{c}}})}{2\pi f_{B_c} m_{B_c}^2} \int_{(m_b+m_c)^2}^{s_0^{B_c}} \text{Im} \Pi_V^{QCD}(s, q^2) \exp\left(\frac{m_{B_c}^2 - s}{M^2}\right) ds,$$

$$\begin{aligned}
A_1(q^2) &= \frac{m_b + m_c}{\pi f_{B_c} m_{B_c}^2 (m_{B_c} + m_{V_{c\bar{c}}})} \int_{(m_b+m_c)^2}^{s_0^{B_c}} \text{Im} \Pi_{A_1}^{QCD}(s, q^2) \exp\left(\frac{m_{B_c}^2 - s}{M^2}\right) ds, \\
A_2(q^2) &= \frac{(m_b + m_c)(m_{B_c} + m_{V_{c\bar{c}}})}{\pi f_{B_c} m_{B_c}^2} \int_{(m_b+m_c)^2}^{s_0^{B_c}} \text{Im} \Pi_{A_2}^{QCD}(s, q^2) \exp\left(\frac{m_{B_c}^2 - s}{M^2}\right) ds, \\
A_3(q^2) &= \frac{(m_b + m_c)(m_{B_c} + m_{V_{c\bar{c}}})}{\pi f_{B_c} m_{B_c}^2} \int_{(m_b+m_c)^2}^{s_0^{B_c}} \text{Im} \Pi_{A_3}^{QCD}(s, q^2) \exp\left(\frac{m_{B_c}^2 - s}{M^2}\right) ds.
\end{aligned} \tag{43}$$

Substituting the QCD representation of the correlation function in Eq.(40) with the help of the OPE technique, we can derive the explicit forms of the form factors in the light-cone QCD sum rules as

$$\begin{aligned}
V(q^2) &= \frac{(m_b + m_c)(m_{B_c} + m_{V_{c\bar{c}}})f_{V_{c\bar{c}}}}{f_{B_c} m_{B_c}^2} \exp\left(\frac{m_{B_c}^2}{M^2}\right) \int_{\Delta}^1 \frac{du}{u} V_T(u) \exp\left[-\frac{m_b^2 - \bar{u}(q^2 - up^2)}{uM^2}\right], \\
A_1(q^2) &= \frac{(m_b + m_c)f_{V_{c\bar{c}}}}{f_{B_c} m_{B_c}^2 (m_{B_c} + m_{V_{c\bar{c}}})} \exp\left(\frac{m_{B_c}^2}{M^2}\right) \int_{\Delta}^1 \frac{du}{u} V_T(u) \exp\left[-\frac{m_b^2 - \bar{u}(q^2 - up^2)}{uM^2}\right] \frac{m_b^2 - q^2 + u^2 p^2}{u}, \\
A_2(q^2) &= -A_3(q^2) = V(q^2),
\end{aligned} \tag{44}$$

with the lower integral limit  $\Delta$  defined by the Eq.(37). It needs to note that similar results were also obtained in Ref. [56, 59].

#### D. Light-cone QCD sum rules for the weak transition form factors of $B_c \rightarrow A_{c\bar{c}}$

The derivation of light-cone QCD sum rules for  $B_c \rightarrow A_{c\bar{c}}$  is very similar to that for  $B_c \rightarrow V_{c\bar{c}}$  discussed before. The correlator for  $B_c \rightarrow A_{c\bar{c}}$  can be given by

$$\Pi_{\mu}(p, q) = i \int d^4x e^{iq \cdot x} \langle A_{c\bar{c}}(p) | T \{ \bar{c}(x) \gamma_{\mu} (1 - \gamma_5) b(x), \bar{b}(0) i(1 + \gamma_5) c(0) \} | 0 \rangle. \tag{45}$$

We will skip the detailed derivation of sum rules for the form factors in  $B_c \rightarrow A_{c\bar{c}}$  and only display the final results of them as

$$\begin{aligned}
A(q^2) &= \frac{(m_b + m_c)(m_{B_c} - m_{A_{c\bar{c}}})f_{A_{c\bar{c}}}}{f_{B_c} m_{B_c}^2} \exp\left(\frac{m_{B_c}^2}{M^2}\right) \int_{\Delta}^1 \frac{du}{u} \phi_{\perp}(u) \exp\left[-\frac{m_b^2 - \bar{u}(q^2 - up^2)}{uM^2}\right], \\
V_1(q^2) &= \frac{(m_b + m_c)f_{A_{c\bar{c}}}}{f_{B_c} m_{B_c}^2 (m_{B_c} - m_{A_{c\bar{c}}})} \exp\left(\frac{m_{B_c}^2}{M^2}\right) \int_{\Delta}^1 \frac{du}{u} \phi_{\perp}(u) \exp\left[-\frac{m_b^2 - \bar{u}(q^2 - up^2)}{uM^2}\right] \frac{m_b^2 - q^2 + u^2 p^2}{u}, \\
V_2(q^2) &= -V_3(q^2) = A(q^2).
\end{aligned} \tag{46}$$

#### IV. NUMERICAL RESULTS FOR THE FORM FACTORS AND DECAY RATES

Now we are going to analyze the sum rules for the form factors numerically. Firstly, we collect the input parameters used in this paper as below [42, 60, 61, 62, 63, 64]:

$$\begin{aligned}
m_e &= 0.511\text{MeV}, & m_\tau &= 1.777\text{GeV}, \\
m_b &= (4.68 \pm 0.03)\text{GeV}, & m_c &= (1.275 \pm 0.015)\text{GeV}, \\
m_{B_c} &= (6.286 \pm 0.005)\text{GeV}, & f_{B_c} &= (395 \pm 15)\text{MeV}, \\
\tau_{B_c} &= (0.463^{+0.073}_{-0.065})\text{ps}, & s_0^{B_c} &= (45 \pm 1)\text{GeV}^2, \\
G_F &= 1.166 \times 10^{-5}\text{GeV}^{-2}, & |V_{cb}| &= (42.21^{+0.10}_{-0.80}) \times 10^{-3}.
\end{aligned} \tag{47}$$

It is noted that the decay constants of various charmonium states have been discussed comprehensively in section II.

The choice of the threshold parameter  $s$  can be determined by the condition that the sum rules should take on the best stability in the allowed  $M^2$  region. Besides, the value of threshold parameter should be around the mass square of the corresponding first excited state, hence they are also chosen the same as that in the usual two-point QCD sum rules. The standard value of the threshold in the  $X$  channel is  $s_{0X} = (m_X + \Delta_X)^2$ , where  $\Delta_X$  is usually taken as  $0.5\text{GeV}$  [65, 66, 67, 68] approximately in the literature. To be more specific, we will adopt the threshold parameter for  $B_c$  channel  $s_0^{B_c}$  as  $(45 \pm 1)\text{GeV}^2$  for the error estimate in the numerical analysis as shown above.

It is well known that the form factors should not depend on the Borel mass  $M$  in the complete theory. However, we can only truncate the operator product expansion up to some finite dimension and perform the perturbative series in  $\alpha_s$  to some order in practice, both of which will result in the dependence of the form factors on the Borel parameter definitely. Therefore, one should find a region where the results only depend moderately on the Borel mass, and the approximations for the above truncations in the complete theory are reasonable and acceptable.

In general, the Borel mass  $M$  should be chosen under the requirement that both the contributions from the higher resonance states and higher twist distribution amplitudes are small (no more than 30 %) to ensure the validity of the OPE near the light-cone and the quark-hadron duality being a good approximation. As for the decay of  $B_c \rightarrow \eta'_c$ , we indeed find the Borel platform  $M^2 \in [20, 30]\text{GeV}^2$ , which is also consistent with the number obtained in the two-point QCD sum rules corresponding to the decay constant of  $f_{B_c}$  [69]. The light-cone QCD sum rules of form factor  $f_+(q^2)$  at zero momentum transfer is shown in Fig. 3. The values of  $f_+(0)$  with various uncertainties rooting in Borel mass, threshold value, decay constants of the related mesons, heavy quark masses and the parameter  $v^2$  involved in the LCDAs of charmonium have been collected in Table I, from which we can find that the total uncertainties

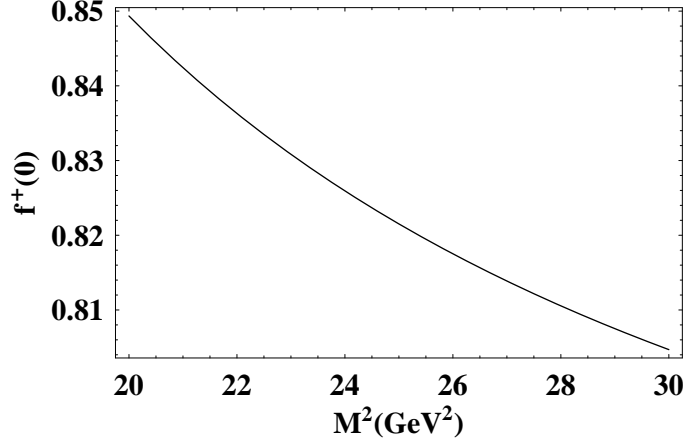


FIG. 3: The form factor  $f_+(0)$  responsible for  $B_c \rightarrow \eta'_c$  decay within the Borel window.

TABLE I: The form factors  $f_i(0)$  responsible for  $B_c \rightarrow P(S)_{c\bar{c}}$  decay in the light-cone QCD sum rules approach; the errors for these entries correspond to the uncertainties in the Borel mass, threshold value, quark masses, decay constants of two mesons and variations of  $v^2$  in the LCDAs of charmonium respectively.

decay modes	$f_+(0)$	$f_-(0)$
$B_c \rightarrow \eta'_c(2^1S_0)$	$0.82^{+0.03+0.02+0.01+0.17+0.01}_{-0.01-0.02-0.01-0.19-0.01}$	0
$B_c \rightarrow X(3940)(3^1S_0)$	$0.46^{+0.01+0.00+0.00+0.10+0.01}_{-0.01-0.01-0.01-0.11-0.01}$	0
$B_c \rightarrow Y(3940)(2^3P_0)$	$2.6^{+0.1+0.0+0.0+0.2+0.0}_{-0.1-0.1-0.1-0.2-0.2}$	0

of form factors are indeed at the level of  $(20 - 30)\%$  as expected by the general understanding of the theoretical framework. The form factor  $f_-(0)$  up to the twist-3 LCDAs of  $P_{c\bar{c}}$  and leading order of  $\alpha_s$  is zero as a result of the large-recoil symmetry. The  $q^2$  dependence of the form factor  $f_+(q^2)$  calculated from light cone sum rules is shown in Fig.4 in the physical kinematical region  $0 \leq q^2 \leq (m_{B_c} - m_{\eta'_c})^2$ . Since the number of  $(m_{B_c} - m_{\eta'_c})^2 \simeq 7.0\text{GeV}^2$  with  $m_{\eta'_c} = 3.638 \pm 0.004\text{GeV}$  [42] being used, is smaller than that of  $(m_b - m_c)^2 - 2\Lambda_{QCD}(m_b - m_c) \simeq 8.2\text{GeV}^2$ , the OPE technique near the light-cone can be performed in the whole kinematical region effectively.

We can further evaluate the sum rules for the form factors associating with  $B_c$  to other charmonium states. For example, the only difference for the calculation of decay mode  $B_c \rightarrow X(3940)(3^1S_0)l\bar{\nu}_l$ , is to substitute the LCDAs of  $X(3940)(3^1S_0)$  for that corresponding to  $\eta'_c$  compared with the decay of  $B_c \rightarrow \eta'_c l\bar{\nu}_l$ . In the light of Eq. (39) and the light-cone distribution amplitudes of scalar charmonium

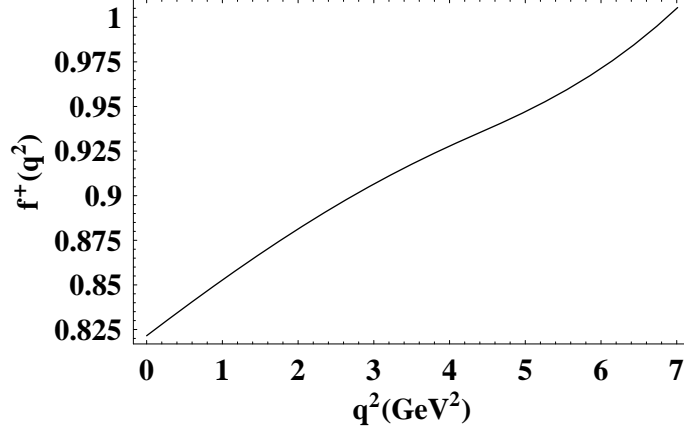


FIG. 4:  $q^2$  dependence of the form factor  $f_+(q^2)$  with  $M^2 = 25\text{GeV}^2$  in the whole physical kinematical region.

calculated before, it's straightforward to estimate the light-cone sum rules for the transition form factors of  $B_c \rightarrow \chi'_{c0} l \bar{\nu}_l$ , the number of which has been grouped in Table I. Evaluations of the form factors relating to the  $B_c \rightarrow V(A)(c\bar{c})$  decay are also easily carried out with the help of Eq.(44,46) and the LCDAs of (axial) vector meson displayed in Section II. Since the calculations for all of these form factors are quite similar, we will not explicitly repeat the details anymore and only display the final results in Table II.

Utilizing the above form factors and the input parameters shown in Eq.(47), we can proceed to compute the branching ratios of these modes. Following the standard procedure, the differential partial decay rate for  $B_c \rightarrow M_{c\bar{c}} l \bar{\nu}_l$  ( $l = e, \tau$ ) can be written as [42]

$$\frac{d\Gamma_{B_c \rightarrow M_{c\bar{c}} l \bar{\nu}_l}}{dq^2} = \frac{1}{(2\pi)^3} \frac{1}{32m_{B_c}^3} \int_{u_{min}}^{u_{max}} |\widetilde{M}_{B_c \rightarrow M_{c\bar{c}} l \bar{\nu}_l}|^2 du, \quad (48)$$

where  $u = (p_{M_{c\bar{c}}} + p_l)^2$  and  $q^2 = (p_l + p_{\bar{\nu}_l})^2$ ;  $p_{M_{c\bar{c}}}$ ,  $p_l$  and  $p_{\bar{\nu}_l}$  are the momenta of  $M_{c\bar{c}}$ ,  $l$  and  $\bar{\nu}_l$  respectively;  $\widetilde{M}$  is the decay amplitude after integrating over the angle between the  $l$  and  $M_{c\bar{c}}$ . The upper and lower limit of  $u$  are given by

$$\begin{aligned} u_{max} &= (E_{M_{c\bar{c}}}^* + E_l^*)^2 - (\sqrt{E_{M_{c\bar{c}}}^{*2} - m_{M_{c\bar{c}}}^2} - \sqrt{E_l^{*2} - m_l^2})^2, \\ u_{min} &= (E_{M_{c\bar{c}}}^* + E_l^*)^2 - (\sqrt{E_{M_{c\bar{c}}}^{*2} - m_{M_{c\bar{c}}}^2} + \sqrt{E_l^{*2} - m_l^2})^2; \end{aligned} \quad (49)$$

where  $E_{M_{c\bar{c}}}^*$  and  $E_l^*$  are the energies of the charmonium state and the lepton in the rest frame of lepton-neutrino pair respectively and the manifest expressions of them can be given by

$$E_{M_{c\bar{c}}}^* = \frac{m_{B_c}^2 - m_{M_{c\bar{c}}}^2 - q^2}{2\sqrt{q^2}}, \quad E_l^* = \frac{q^2 + m_l^2}{2\sqrt{q^2}}. \quad (50)$$



TABLE II: The form factors  $f_i(0)$  responsible for  $B_c \rightarrow V_{c\bar{c}}(A_{c\bar{c}})$  decays in the light-cone QCD sum rules approach; the errors for these entries correspond to the uncertainties in the Borel mass, threshold value, quark masses, decay constants of two mesons and variations of  $v^2$  in the LCDAs of charmonium respectively.

decay mode	$V(0)$	$A_1(0)$
$B_c \rightarrow \psi(2^3S_1)$	$0.90^{+0.03+0.02+0.02+0.04+0.01}_{-0.02-0.03-0.02-0.05-0.00}$	$0.38^{+0.01+0.01+0.00+0.02+0.00}_{-0.01-0.02-0.01-0.02-0.00}$
$B_c \rightarrow \psi(1^3D_1)$	$0.11^{+0.00+0.01+0.00+0.00+0.00}_{-0.00-0.01-0.00-0.00-0.01}$	$4.9^{+0.0+0.5+0.2+0.2+0.1}_{-0.1-0.6-0.3-0.2-0.3} \times 10^{-2}$
$B_c \rightarrow \psi(3^3S_1)$	$0.52^{+0.02+0.00+0.00+0.04+0.00}_{-0.01-0.01-0.01-0.04-0.00}$	$0.21^{+0.01+0.00+0.00+0.02+0.00}_{-0.01-0.00-0.00-0.02-0.00}$
$B_c \rightarrow \psi(2^3D_1)$	$7.2^{+0.0+0.9+0.5+0.4+0.6}_{-0.1-1.0-0.6-0.5-0.6} \times 10^{-2}$	$3.0^{+0.0+0.4+0.2+0.2+0.3}_{-0.0-0.4-0.2-0.2-0.2} \times 10^{-2}$
$B_c \rightarrow Y(4260)(4^3S_1)$	$0.47^{+0.01+0.00+0.00+0.07+0.00}_{-0.01-0.01-0.01-0.07-0.01}$	$0.18^{+0.01+0.01+0.00+0.03+0.01}_{-0.00-0.00-0.00-0.03-0.0}$
$B_c \rightarrow \psi(3^3D_1)$	$3.8^{+0.0+0.6+0.3+0.3+0.4}_{-0.1-0.7-0.4-0.3-0.5} \times 10^{-2}$	$1.5^{+0.1+0.3+0.2+0.1+0.2}_{-0.0-0.2-0.1-0.1-0.1} \times 10^{-2}$
	$A(0)$	$V_1(0)$
$B_c \rightarrow X(3872)(2^3P_1)$	$-0.53^{+0.02+0.01+0.01+0.02+0.04}_{-0.03-0.00-0.00-0.02-0.03}$	$-3.76^{+0.12+0.05+0.02+0.14+0.24}_{-0.18-0.02-0.00-0.14-0.20}$
$B_c \rightarrow X(3940)(2^3P_1)$	$-0.51^{+0.02+0.01+0.01+0.02+0.04}_{-0.02-0.00-0.01-0.02-0.03}$	$-3.87^{+0.11+0.05+0.02+0.15+0.24}_{-0.19-0.03-0.02-0.15-0.22}$
$B_c \rightarrow h_c(1^1P_1)$	$0.28^{+0.01+0.01+0.01+0.01+0.00}_{-0.01-0.00-0.00-0.01-0.00}$	$1.51^{+0.04+0.04+0.02+0.06+0.01}_{-0.02-0.04-0.02-0.06-0.00}$
$B_c \rightarrow h_c(2^1P_1)$	$0.14^{+0.00+0.00+0.00+0.01+0.00}_{-0.00-0.01-0.00-0.01-0.00}$	$1.10^{+0.00+0.00+0.04+0.00+0.00}_{-0.00-0.00-0.04-0.00-0.00}$

The numerical results are shown in Table III and IV, together with the numbers obtained in other approaches for comparison. It is observed that the decay rates for  $B_c \rightarrow h_c$  and  $B_c \rightarrow \eta'_c$  calculated in this work are consistent with that obtained in other frameworks [61, 70, 71, 72, 73] within the error bars, such as quark model, Bethe-Salpeter equation, SVZ sum rules and so on, therefore, the branching fractions for  $B_c$  to the new charmonium states presented in this work are reliable and acceptable.

It should be noted that the assignment of  $X(3940)$  as  $3^1S_0$  charmonium state leads to the production rate as  $1.9 \times 10^{-4}$  in the weak decay of  $B_c \rightarrow X(3940)e\bar{\nu}_e$ , the magnitude of which is one order smaller than that for the interpretation of  $X(3940)$  being a  $2^3P_1$  charmonium. This particular phenomenology can provide valuable information for us to discover the inner structures of  $X(3940)$ . Besides, both of  $X(3872)$  and  $Y(3940)$  should be observed in the weak decay of  $B_c \rightarrow X(3872)/Y(3940)e\bar{\nu}_e$  in view of the branching ratio as large as  $10^{-3}$  order, on the condition that they can be explained as  $2^3P_1$  and  $2^3P_0$  charmonium states respectively. If future experimental measurements deviate from our predictions heavily, it will rule out the current  $J^{PC}$  assignments of the charmonium states. It also needs to mention that the decay rates for semi-leptonic decays of  $B_c \rightarrow M_{c\bar{c}}\tau\bar{\nu}_\tau$  are also displayed in Table IV, from which we can find that they are about one order smaller than the corresponding channel  $B_c \rightarrow M_{c\bar{c}}e\bar{\nu}_e$  due to suppression from the phase space and sensitive dependence of form factors on the momentum transfer  $q^2$ . In particular, the branching fraction of  $B_c \rightarrow Y(4260)\tau\bar{\nu}_\tau$  is about two orders smaller than that for

TABLE III: Branching fractions of  $B_c \rightarrow M_{c\bar{c}} e \bar{\nu}_e$  semi-leptonic decays in the light-cone QCD sum rules approach; the errors for these entries correspond to the uncertainties in the Borel mass, threshold value, quark masses, decay constants of two mesons, lifetime of  $B_c$  and variations of  $v^2$  in the LCDAs of charmonium respectively.

decay modes	BR(this work)	Other works
$B_c \rightarrow \eta'_c(2^1S_0)e\bar{\nu}_e$	$1.1^{+0.1+0.0+0.0+0.4+0.4+0.0}_{-0.0-0.0-0.0-0.5-0.4-0.0} \times 10^{-3}$	$3.2 \times 10^{-4}$ [70] $5.1 \times 10^{-4}$ [71]
$B_c \rightarrow X(3940)(3^1S_0)e\bar{\nu}_e$	$1.9^{+0.2+0.1+0.0+0.8+0.7+0.0}_{-0.1-0.1-0.0-0.9-0.7-0.0} \times 10^{-4}$	
$B_c \rightarrow Y(3940)(2^3P_0)e\bar{\nu}_e$	$7.2^{+0.9+0.1+0.1+0.9+2.5+0.0}_{-0.5-0.0-0.0-0.9-2.8-0.9} \times 10^{-3}$	
$B_c \rightarrow Y(4260)(4^3S_1)e\bar{\nu}_e$	$1.5^{+0.1+0.0+0.1+0.1+0.5+0.0}_{-0.1-0.1-0.0-0.1-0.6-0.0} \times 10^{-4}$	
$B_c \rightarrow X(3872)(2^3P_1)e\bar{\nu}_e$	$6.7^{+0.9+0.0+0.1+0.5+2.3+0.7}_{-0.5-0.0-0.0-0.5-2.6-0.7} \times 10^{-3}$	
$B_c \rightarrow X(3940)(2^3P_1)e\bar{\nu}_e$	$6.0^{+0.7+0.0+0.0+0.5+2.1+0.6}_{-0.5-0.1-0.1-0.5-2.3-0.7} \times 10^{-3}$	
$B_c \rightarrow h_c(1^1P_1)e\bar{\nu}_e$	$2.9^{+0.3+0.0+0.0+0.2+1.0+0.1}_{-0.1-0.0-0.0-0.2-1.1-0.0} \times 10^{-3}$	$2.7 \times 10^{-3}$ [61] $1.7^{+0.2}_{-0.0} \times 10^{-3}$ [72]
$B_c \rightarrow h'_c(2^1P_1)e\bar{\nu}_e$	$5.3^{+0.3+0.1+0.0+0.4+1.8+0.1}_{-0.2-0.1-0.0-0.4-2.1-0.0} \times 10^{-4}$	

TABLE IV: Branching fractions of  $B_c \rightarrow M_{c\bar{c}} \tau \bar{\nu}_\tau$  semi-leptonic decays in the light-cone QCD sum rules approach; the errors for these entries correspond to the uncertainties in the Borel mass, threshold value, quark masses, decay constants of two mesons, lifetime of  $B_c$  and variations of  $v^2$  in the LCDAs of charmonium respectively .

decay modes	BR(this work)	Other works
$B_c \rightarrow \eta'_c(2^1S_0)\tau\bar{\nu}_\tau$	$8.1^{+0.9+0.1+0.1+3.3+2.8+0.1}_{-0.5-0.1-0.1-3.7-3.2-0.0} \times 10^{-5}$	$1.6 \times 10^{-5}$ [73]
$B_c \rightarrow X(3940)(3^1S_0)\tau\bar{\nu}_\tau$	$5.7^{+0.6+0.7+0.3+2.4+2.0+0.0}_{-0.3-0.4-0.3-2.7-2.2-0.1} \times 10^{-6}$	
$B_c \rightarrow Y(3940)(2^3P_0)\tau\bar{\nu}_\tau$	$2.7^{+0.4+0.0+0.0+0.3+0.9+0.0}_{-0.2-0.0-0.0-0.3-1.1-0.3} \times 10^{-4}$	
$B_c \rightarrow Y(4260)(4^3S_1)\tau\bar{\nu}_\tau$	$6.4^{+0.5+0.8+0.3+0.5+2.2+0.1}_{-0.3-0.4-0.2-0.5-2.5-0.0} \times 10^{-7}$	
$B_c \rightarrow X(3872)(2^3P_1)\tau\bar{\nu}_\tau$	$3.2^{+0.5+0.0+0.0+0.2+1.1+0.4}_{-0.2-0.2-0.0-0.2-1.3-0.3} \times 10^{-4}$	
$B_c \rightarrow X(3940)(2^3P_1)\tau\bar{\nu}_\tau$	$2.2^{+0.3+0.0+0.1+0.2+0.8+0.2}_{-0.2-0.0-0.0-0.2-0.9-0.3} \times 10^{-4}$	
$B_c \rightarrow h_c(1^1P_1)\tau\bar{\nu}_\tau$	$3.7^{+0.4+0.1+0.1+0.3+1.3+0.1}_{-0.2-0.1-0.0-0.3-1.4-0.0} \times 10^{-4}$	$1.7 \times 10^{-4}$ [61] $1.5^{+0.1}_{-0.0} \times 10^{-4}$ [72]
$B_c \rightarrow h'_c(2^1P_1)\tau\bar{\nu}_\tau$	$2.0^{+0.2+0.0+0.0+0.2+0.7+0.1}_{-0.1-0.0-0.0-0.2-0.8-0.0} \times 10^{-5}$	

the  $B_c \rightarrow Y(4260)e\bar{\nu}_e$  mode, since the sum of the mass for  $Y(4260)$  and  $\tau$  lepton is almost close to the threshold of  $B_c$  meson.

Finally, we are in a position of concentrating on the  $S - D$  mixing of various vector charmonium. It's known that the  $S - D$  mixing of  $\psi(3686)$  and  $\psi(3770)$  may be essential to explain the large leptonic

decay width of  $\psi(3770)$ , the notorious  $\rho\pi$  puzzle [47] and the enhancement of  $\psi(3686) \rightarrow K_L K_S$  [74]. The production of  $\psi(3770)$  in  $B$  meson decays  $B^+ \rightarrow \psi(3770)K^+$  is found to be surprisingly large by Belle [75], which can be even comparable to  $B^+ \rightarrow \psi(3686)K^+$  [76, 77]. Hence, it is helpful to investigate the weak production of  $\psi(3686)$  and  $\psi(3770)$  in  $B_c$  decays in order to test the above mixing scheme further and clarify the inner structures of them. Assuming that the physical state  $\psi(3686)$  and  $\psi(3770)$  are the mixture of  $1^3D_1$  and  $2^3S_1$  states, we have

$$\begin{aligned} |\psi(3686)\rangle &= \cos\theta|2^3S_1\rangle + \sin\theta|1^3D_1\rangle, \\ |\psi(3770)\rangle &= -\sin\theta|2^3S_1\rangle + \cos\theta|1^3D_1\rangle. \end{aligned} \quad (51)$$

As for the mixing angle  $\theta$ , two solutions  $\theta = -(12 \pm 2)^\circ$  or  $\theta = (27 \pm 2)^\circ$  [78, 79], were found in order to reproduce the leptonic widths of  $\psi(3686)$  and  $\psi(3770)$  [47]. The small mixing solution, i.e.,  $\theta = -(12 \pm 2)^\circ$ , is consistent with couple-channel estimates [80, 81] and the E1 transition  $\psi' \rightarrow \chi_{cJ}\gamma$  [79].

Based on the transition form factors of  $B_c \rightarrow \psi(2^3S_1)$  and  $B_c \rightarrow \psi(1^3D_1)$  listed in Table II, we can plot the production rates of them in the  $B_c$  decays as functions of the mixing angle  $\theta$ , which are displayed in Fig. 5. As for the favored mixing angle  $\theta = -12^\circ$ , the branching fraction of  $B_c \rightarrow \psi(3686)e\bar{\nu}_e$  is  $1.5 \times 10^{-3}$ , which is almost the same as the number of  $1.7 \times 10^{-3}$  in the case of  $2^3S_1$  state without mixing. However, the mixing component of  $\psi(2^3S_1)$  in the structure of  $\psi(3770)$  is in particular important, which can increase the decay rate of  $B_c \rightarrow \psi(3770)e\bar{\nu}_e$  from  $4.5 \times 10^{-5}$  to  $2.1 \times 10^{-4}$ . The reason is that the decay constant of  $1^3D_1$  charmonium (47.8 MeV) is too small compared with that of  $2^3S_1$  state (304 MeV), therefore, even a small mixing angle can affect the decay rate of  $B_c \rightarrow \psi(3770)e\bar{\nu}_e$  drastically, but almost has no effect on the decay  $B_c \rightarrow \psi(3686)e\bar{\nu}_e$ .

Besides, exploring the properties of  $Y(4260)$  and  $\psi(4415)$  as the mixing of  $4S$  and  $3D$  states in the weak decays can also shed light on the inner structures of these charmonium-like particles. On the one hand, the assignment of  $3^3D_1$  charmonium for  $\psi(4415)$  is supported by the fact that  $\psi(4415)$  is dominated by the decay of  $\psi(4415) \rightarrow D\bar{D}_2^*(2460)$  reported by the Belle Collaboration very recently [82]. On the other hand,  $Y(4260)$  can be accommodated as  $4S$  state naturally based on the analysis of production and decay characters of it [21]. Moreover, the absence of  $Y(4260)$  signal in  $e^+e^- \rightarrow \text{hadrons}$  can be explained quantitatively in the  $S-D$  mixing scheme [83]. Similar to the  $S-D$  mixing of  $\psi(3686)$  and  $\psi(3770)$ , we express the states of  $Y(4260)$  and  $\psi(4415)$  as

$$\begin{aligned} |Y(4260)\rangle &= \cos\theta|4^3S_1\rangle + \sin\theta|3^3D_1\rangle, \\ |\psi(4415)\rangle &= -\sin\theta|4^3S_1\rangle + \cos\theta|3^3D_1\rangle. \end{aligned} \quad (52)$$

In the above mixing picture, we can analyze the dependence of production rates for  $Y(4260)$  and  $\psi(4415)$  in the weak  $B_c$  decays on the mixing angle  $\theta$  with the help of the form factors of  $B_c \rightarrow \psi(4^3S_1)$

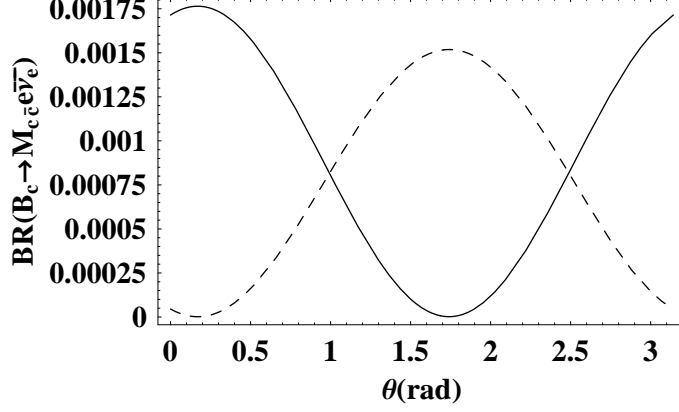


FIG. 5: Decay rates of  $B_c \rightarrow \psi(3686)e\bar{\nu}_e$  and  $B_c \rightarrow \psi(3770)e\bar{\nu}_e$  as functions of the mixing angle  $\theta$ . The solid line represents the case of  $B_c \rightarrow \psi(3686)e\bar{\nu}_e$ , while the dashed line is for  $B_c \rightarrow \psi(3770)e\bar{\nu}_e$ .

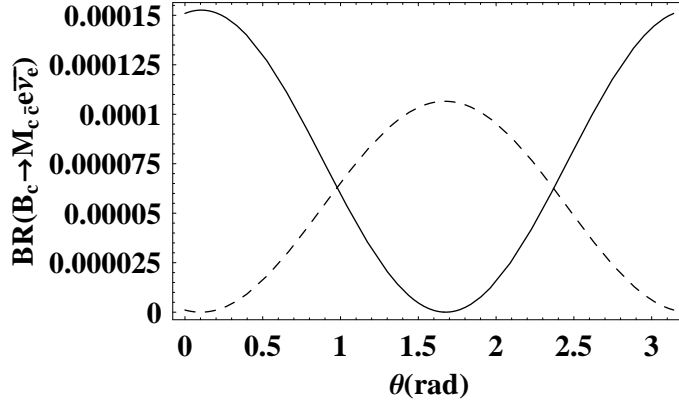


FIG. 6: Decay rates of  $B_c \rightarrow Y(4260)e\bar{\nu}_e$  and  $B_c \rightarrow \psi(4415)e\bar{\nu}_e$  as functions of the mixing angle  $\theta$ . The solid line represents the case of  $B_c \rightarrow Y(4260)e\bar{\nu}_e$ , while the dashed line is for  $B_c \rightarrow \psi(4415)e\bar{\nu}_e$ .

and  $B_c \rightarrow \psi(3^3D_1)$  calculated before. As shown in Fig. 6, it can be observed that the decay rate for  $B_c \rightarrow Y(4260)e\bar{\nu}_e$  and  $B_c \rightarrow \psi(4415)e\bar{\nu}_e$  are  $1.5 \times 10^{-4}$  and  $1.1 \times 10^{-6}$  for the null mixing angle. As a simple test, we find that the production rate of  $\psi(4415)$  in  $B_c$  decay can reach as large as  $1.0 \times 10^{-5}$  for the mixing angle of  $\theta = -12^\circ$ ; while the branching fraction of  $B_c \rightarrow Y(4260)e\bar{\nu}_e$  is  $1.4 \times 10^{-4}$ , almost the same as that in the case with zero mixing angle.

For the completeness, we also consider the  $\psi(4040)$  and  $\psi(4160)$  being the mixing states of  $3S$  and

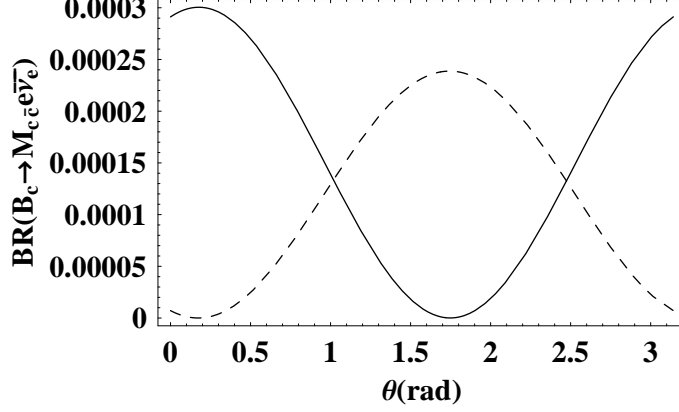


FIG. 7: Decay rates of  $B_c \rightarrow \psi(4040)e\bar{\nu}_e$  and  $B_c \rightarrow \psi(4160)e\bar{\nu}_e$  as functions of the mixing angle  $\theta$ . The solid line represents the case of  $B_c \rightarrow \psi(4040)e\bar{\nu}_e$ , while the dashed line is for  $B_c \rightarrow \psi(4160)e\bar{\nu}_e$ .

2D as

$$\begin{aligned}
|\psi(4040)\rangle &= \cos\theta|3^3S_1\rangle + \sin\theta|2^3D_1\rangle, \\
|\psi(4160)\rangle &= -\sin\theta|3^3S_1\rangle + \cos\theta|2^3D_1\rangle.
\end{aligned} \tag{53}$$

The dependence of production rates for  $B_c \rightarrow \psi(4040)e\bar{\nu}_e$  and  $B_c \rightarrow \psi(4160)e\bar{\nu}_e$  on the mixing angle  $\theta$  have been plotted explicitly in Fig. 7. Without the mixing of  $S - D$  states, the branching ratios for  $B_c \rightarrow \psi(4040)e\bar{\nu}_e$  and  $B_c \rightarrow \psi(4160)e\bar{\nu}_e$  are  $2.9 \times 10^{-4}$  and  $7.1 \times 10^{-6}$  respectively. As for the mixing angle  $\theta = -12^\circ$ , the decay rates become  $2.6 \times 10^{-4}$  and  $3.3 \times 10^{-5}$  for  $B_c \rightarrow \psi(4040)e\bar{\nu}_e$  and  $B_c \rightarrow \psi(4160)e\bar{\nu}_e$ , from which we can observe a large enhancement for the production of  $\psi(4160)$  as we expect.

Up to now, we only focus on the discussions of  $S - D$  mixing for the decays of  $B_c \rightarrow V_{c\bar{c}}e\bar{\nu}_e$ , which can be readily generalized to the case for the  $B_c \rightarrow V_{c\bar{c}}\tau\bar{\nu}_\tau$  decays. Subsequently, similar observations to the final states being the  $e\bar{\nu}_e$  pair can be achieved: Even a small mixing angle can result in considerable effects on the decay rates of  $B_c \rightarrow \psi(n^3D_1)\tau\bar{\nu}_\tau$  ( $n = 1, 2, 3$ ), while branching fractions of the corresponding channels  $B_c \rightarrow \psi((n+1)^3S_1)\tau\bar{\nu}_\tau$  do not vary significantly.

## V. SUMMARY

A number of new heavy charmonium states, such as  $\eta'_c$ ,  $h_c$ ,  $X(3940)$ ,  $Y(3940)$ ,  $X(3872)$  and  $Y(4260)$  are observed during the past several years. There exist various explanations for the quark components

of the heavy mesons  $X(3940)$ ,  $Y(3940)$ ,  $X(3872)$ ,  $Y(4260)$  so far, such as charmonium states, tetraquark pictures, molecular bound states and so on. It is still early to give a definite answer for their solutions.

In this work, we mainly focus on the charmonium interpretation of all the states,  $\eta'_c$ ,  $h_c$ ,  $h'_c$ ,  $X(3940)$ ,  $Y(3940)$ ,  $X(3872)$ , and  $Y(4260)$  produced in the exclusive semi-leptonic weak decay of  $B_c$  meson. In order to compute the branching ratios of semi-leptonic weak decays of  $B_c$ , we need to deal with the hadronic transition matrix element  $\langle M_{c\bar{c}} | j_\mu | B_c \rangle$ , which defines the form factors governed mainly by non-perturbative QCD effects. In this paper, the light-cone QCD sum rules approach is used to evaluate various form factors. We choose the correlation function with the insertion of chiral current following the Ref. [51, 52], the consequences of which are that the twist-3 LCDAs do not contribute to the sum rules for  $B_c$  decays to the pseudoscalar charmonium and also for  $B_c$  decays to the (axial) vector charmonium in the absence of three-particle wave functions.

With the help of form factors calculated in the light-cone QCD sum rules approach, we give the decay rates for semi-leptonic decays of  $B_c \rightarrow h_c$  and  $\eta'_c$ , which agree with that derived in other frameworks. Besides, it is found that different interpretations of  $X(3940)$  can result in remarkable difference of the production rate in the  $B_c$  decays, which would help to clarify the quark structures of the  $X(3940)$  with the forthcoming LHC-b experiments. Furthermore, the weak productions of  $X(3872)$  and  $Y(3940)$  in  $B_c$  decays are large enough to be detected in the future experiments, supposing that they are indeed  $2^3P_1$  and  $2^3P_0$  charmonium states respectively. It is also observed that the mixing component of  $2^3S_1$  charmonium state in the structure of  $\psi(3770)$  can enhance its production rate in  $B_c$  decays heavily, even for a small mixing angle. Besides, the production character of  $Y(4260)$  and  $\psi(4415)$  as the mixing of  $4S$  and  $3D$  states as well as  $\psi(4040)$  and  $\psi(4160)$  being the mixing states of  $3S$  and  $2D$  are also included in this work. In fact, all these decay rates depend heavily on the  $J^{PC}$  assignments of the charmonium states. Therefore, our calculations can be used in LHC-b experiment to explore the components of these hidden charm mesons.

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## APPENDIX A: AN EXAMPLE OF CONSTRUCTING LCDAS FOR CHARMONIUM STATES

Taking  $\eta_c''$  meson as an example, we would like to explain the construction of LCDAs for heavy quarkonium step by step in this appendix based on the procedure [40, 45] described in section II.

Firstly, we write down the radial Schrödinger wavefunction of  $n = 3$ ,  $l = 0$  state for the Coulomb potential as

$$\psi_{Sch}(r) \propto [1 - \frac{2}{3}q_B r + \frac{2}{27}(q_B r)^2] \exp(-\frac{q_B r}{3}), \quad (A1)$$

where  $q_B$  is the Bohr momentum. Performing the Fourier transformation of the above wavefunction, then we can arrive at

$$\psi_{Sch}(k) \propto \frac{q_B^4 - 30q_B^2 k^2 + 81k^4}{(9k^2 + q_B^2)^4}, \quad (A2)$$

with  $k^2$  being the square of three momentum, namely  $k^2 = |\mathbf{k}|^2$ . In terms of the substitution assumption [46]

$$\mathbf{k}_\perp \rightarrow \mathbf{k}_\perp, \quad k_z \rightarrow (2x - 1)\frac{m_0}{2}, \quad m_0^2 = \frac{m_c^2 + \mathbf{k}_\perp^2}{x(1 - x)}. \quad (A3)$$

we should make the following replacement towards the variable  $k^2$

$$k^2 \rightarrow \frac{\mathbf{k}_\perp^2 + (1 - 2x)^2 m_c^2}{4x(1 - x)}. \quad (A4)$$

Now, we can derive the Schrödinger wavefunction for  $\eta_c''$  as

$$\begin{aligned} \psi_{Sch}(x) &\propto \int d^2 \mathbf{k}_\perp \psi_{Sch}(x, \mathbf{k}_\perp) \\ &\propto x(1 - x) \left\{ \frac{x(1 - x)[1 - 4x(1 - x)(1 + \frac{v^2}{27})]^2}{[1 - 4x(1 - x)(1 - \frac{v^2}{9})]^3} \right\}, \end{aligned} \quad (A5)$$

where  $v^2$  is defined as  $v^2 = q_B^2/m_c^2$ . Following the Ref. [40, 45], we propose the LCDAs of  $\eta_c''$  as

$$\phi^{v,s}(x) = \phi_{asy}^{v,s}(x) \left\{ \frac{x(1 - x)[1 - 4x(1 - x)(1 + \frac{v^2}{27})]^2}{[1 - 4x(1 - x)(1 - \frac{v^2}{9})]^3} \right\}^{1-v^2}, \quad (A6)$$

where the power  $1 - v^2$  reflects the small relativistic corrections to the Coulomb wavefunctions [45]. It can be observed that above distribution amplitudes have the correct asymptotic behavior for both the heavy quarkonium in the heavy quark limit  $v^2 \rightarrow 0$  and light mesons in the  $v^2 \rightarrow 1$  limit.

Moreover, it is known that the asymptotic forms of pseudoscalar mesons can be given by

$$\phi_{asy}^v(x) \propto x(1 - x), \quad \phi_{asy}^s(x) \propto 1. \quad (A7)$$

Substituting Eq. (A7) into Eq. (A5), we can obtain the LCDAs for  $\eta_c''$

$$\begin{aligned}\phi^v(x) &= 10.8x(1-x) \left\{ \frac{x(1-x)[1-4x(1-x)(1+\frac{v^2}{27})]^2}{[1-4x(1-x)(1-\frac{v^2}{9})]^3} \right\}^{1-v^2}, \\ \phi^s(x) &= 2.1 \left\{ \frac{x(1-x)[1-4x(1-x)(1+\frac{v^2}{27})]^2}{[1-4x(1-x)(1-\frac{v^2}{9})]^3} \right\}^{1-v^2},\end{aligned}\tag{A8}$$

corresponding to the given value of  $v^2 = 0.30$  as displayed in the text, where the normalization condition  $\int_0^1 \phi^{v,s}(x)dx = 1$  has been used in the derivation of above LCDAs. In addition, we find that the fluctuations of the phenomenological parameter  $v^2$  do not have significant effects on the shape of distribution amplitudes for charmonium states generally within the acceptable range  $v^2 = 0.30 \pm 0.05$ , which can also be verified from the numbers of decay rate for  $B_c \rightarrow M_{cc}\bar{l}\bar{\nu}_l$  grouped in Table III and IV.

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